THE MATHEMATICS TEACHER

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VOLUME VII JUNE, 1915 NUMBER 4

REVERSAL EQUATIONS.

By H. T. Merritt.

In the September issue of this magazine Prof. W. R. Ransom has called attention in his article on "The Mathematical Pessimist" to a class of equations not recognized in our algebras, and says in this connection: "I consider that in calling attention to the claim of reversal equations to recognition as a *class* of equal importance with linears and quadratics, I have made my chief contribution to algebraic progress."

May I submit that aside from the claim for recognition as a *class*, the reversal equation is worthy of consideration as a *method?* There is a point of view in their solution that is perhaps novel, and certainly helpful in some cases. For several years I have made continual use of this method with review and advanced classes and I am convinced that much can be done with it in beginning classes, although I have never had an opportunity to give it a thorough trial there. The especial value of the method is that it is strongest where the ordinary method is weakest, or at least most troublesome to the beginner; that is, in literal equations and in the transformation of formulas and functional expressions. Anything that will help dispel the phobia that strikes the pupil when he first comes to the requirement, "Solve $A = \frac{1}{2}h(b + b')$ for b," will, I feel sure, be of interest to some teachers. There is here no panacea, but there is something that assures the sufferer that the medicine that he is taking is for a definite purpose.

The reversal method applies to all equations in which the unknown appears only once. This is not so great a restriction as appears, for every equation met in elementary algebra may be reduced to this form. The idea back of the solution of a reversal is to consider what has been done to the unknown in order to *involve* it in the particular manner indicated by the equation, and then by reversing the things done in reverse order, *evolve* the unknown from the equation. The key to the solution is to answer the question, "What has been done to the quantity for which we wish to obtain a value?" For example consider the equation

$$\sqrt{\frac{(3x-4)^3}{2}+5}=3.$$

To solve, the question "What has been done to x?" must be answered first. Note that this cannot be done by pupils whose knowledge of the meaning of the symbols used is at all hazy and that therefore a premium is put on the accurate interpretation of the forms used. Pupils do interpret from the inside outward easily, however, and the answers to this self-asked question are readily and correctly given. In this instance, x is first multiplied by 3, then subtract 4, raise to the third power, divide by 2, add 5 and take the square root, obtaining 3. There now remains the reversal of these operations in the reverse order. We start with the thing obtained, 3, square, subtract 5, multiply by 2, find the cube root, add 4 and divide by 3.

This can be put into compact and convenient form as follows:

$$x = M/3,\ S/4,\ P/3,\ D/2,\ A/5,\ R/2 = 3,$$

$$x = \frac{\sqrt[3]{2(3^2-5)}+4}{3} = 2.$$

As an illustration of a literal, solve for R

$$S = \pi r\sqrt{4R^2 - r^2},$$

$$R = P/2,\ M/4,\ S/r^2,\ R/2,\ M/\pi r = S,$$

$$R = \sqrt{\frac{\left(\frac{S}{\pi r}\right)^2 + r^2}{4}} = \frac{1}{2\pi r}\sqrt{S^2+\pi^2 r^4}.$$

It is in the solution of literal equations of this type and in the transformation of the many formulas of geometry and physics that the pupil will find the reversal method most helpful. The newer texts are laying greater emphasis on the importance of work of this kind, yet the impression prevails among pupils, if not among teachers, that such examples are nothing but a species of algebraic puzzle. I believe that this idea has come from the methods used in solving. The pupil who dares attack an involved expression to solve for a letter contained in it adopts a sort of catch-as-catch-can method and endeavors to pin the shoulders of the unknown to the mat before his own are there. If there is a strategy that he can employ to apparently gain his end, he does not hesitate to adopt it, no matter how doubtful it may be. In the reversal method, however, the plan of attack is definitely laid out. Like Theseus he proceeds against the enemy, laying a trail behind him, in full confidence that the steps can be retraced and that he will come out the victor. The assurance that pupils come to have in their ability to cope successfully with this kind of equation is in itself sufficient ground for considering the method.

Leaving for a little the consideration of the special cases in which the method needs amplification, let us compare it with the method of solving linears commonly used. It will at once be said that in reality there is little new here, as the steps taken to obtain the value are in the end the very steps taken in the usual method. If the reversed steps in the first equation given are performed on both members of the given equation, the equation will then be solved by the usual method. But there is, nevertheless, a great difference to the pupil, as in the reversal method the steps are taken over a return route and are not into a strange territory. There is no better evidence of the difference between the methods than the fact that in the usual method there must be a pause after each step in order to see if the right direction has been maintained. In other words the reversal method has whatever advantages the analytic method of attack has in geometry.

Some six years ago I had occasion to take a class of beginners in algebra and I found early in the work that the class had a most delightfully curious and inquisitive state of mind. They

were not, as a class, at all content to learn a certain operation because it would produce a certain result. I was in perfect sympathy with the text we were using in the effort it made to postpone the word *transposition* and the mechanical part of the process until after the reasons for the step were well understood. But I found that the class was not at all content with the explanation of the change in sign by means of the addition or subtraction axiom. They seemed to get as much satisfaction out of the stock illustrations of the balances, and so on, as most of us do, and they also saw that the addition or subtraction did the work, but still something seemed to be lacking. The better pupils apparently felt that there was a wheel in the train somewhere that was turning, but which they had never seen. Then one day their difficulties vanished as the result of a chance illustration. A long strip of carpet which I had just before had occasion to roll up gave the idea that the quantities of an equation might be compared to objects that would be rolled into the strip. To get out the objects rolled in, the carpet must be unrolled and the objects come out in the reverse order from which they went in. So an unknown which has been *involved*, by any of the six simple operations that the pupil can then command, may be *evolved* by undoing the steps in reverse order. The idea met with an instant response; a few similar illustrations were brought up, such as the blazing of a trail into unknown country, taking an unfamiliar machine apart and putting it together again; and the class had mastered a new idea. This way of looking at the equation satisfied their minds where the axioms did not.

With those excellent principles, the axioms, I have no quarrel, but a boy in one of my classes recently defined them as "something that you say when you can't get out of a thing in any other way," and I suspect his attitude of mind fairly represented that of the class and perhaps reflected in a measure the opinion of his teacher. In the solution of equations the axioms give a good explanation of what has happened after it is all over, but are of no use in telling what to do. The reversal method X-rays the equation and defines the method of attack.

It is obvious that the key question, "What has been done to x?" cannot be answered in either of two cases, one where there

is more than one x and the other where x is an operator. As for the first, the method works after the form of the equation is changed to one with a single x, a step as necessary for the solution by any other method as by the reversal. In the case where x is an operator (the exponentials being not considered) there are two possibilities, x as a divisor and as a subtrahend. The latter is easily disposed of by making the first step a multiplication by -1, as, solve for a

$$s = \frac{rl - a}{r - 1},$$

$$a = M/(-1),\ A/rl,\ D/r - 1 = s,$$

$$a = \frac{(s(r-1) - rl)}{-1} = rl - s(r-1).$$

The case where the unknown is a divisor is the most troublesome, yet even here there may be advantages. The obvious solution is to work with reciprocals, either of the unknown itself, if it occurs as a monomial divisor, or of the polynomial expression in which it occurs as a divisor. An alternative is to place the fraction containing the unknown alone in one member and then take the reciprocal of both members. For example:

$$b = \left(\frac{c}{a-x}\right)^2 + k.$$

Change to

$$\frac{1}{b-k} = \left(\frac{a-x}{c}\right)^2.$$

Then

$$x = M/(-1),\ A/a,\ D/c,\ P/2 = \frac{1}{b-k},$$

$$x = \left[\left(\pm\sqrt{\frac{1}{b-k}}\right)\cdot c - a\right](-1) = \pm c\sqrt{\frac{1}{b-k}} + a.$$

This last example is an interesting one, in that the average pupil may make a bad mess of it by the ordinary method. He will probably expand and clear of fractions as a matter of habit. He will then probably miss the simple trinomial square that would save the day. The solution by formula will then be

chosen, and if the form simplifies, it is a matter for congratulation.

Perhaps the most valuable consideration in connection with the reversal method is that it is more a habit of mind than a method, in its last analysis. Anyone who has a mastery of algebraic processes uses the main points of the method unconsciously, if not consciously. The looking beneath the surface of an expression before doing anything with it is a habit that we try to instil in the early stages of the study. If there is in this method, as here briefly outlined, any suggestion that will help in this direction, it will have served its purpose.

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RECONSTRUCTED MATHEMATICS IN THE HIGH SCHOOL.*

By Henry C. Morrison.

Few are satisfied with the present mathematics situation in the high school, particularly in the first two years of the high school. Dissatisfaction is found in the college faculties which deal with the product, among the mathematicians who are looking for a foundation for productive scholarship, among the teachers who are looking for something better; and dissatisfaction coupled with ridicule is found among the business men, engineers, and others, who expect mathematics learned in the school to function in the practical affairs of life. The purpose of this paper is an attempt to analyze the situation, to find out what is the matter with mathematics in the high school, and if possible to throw some light on the way out.

Three Factors Involved: Students, Social Needs, Available Subject-Matter.—Whatever the solution ultimately may be found to be, it can confidently be stated that the three chief factors of the problem to be solved here, as in the case of all other curriculum problems, are: (1) the pupil and the laws of his mental growth and development; (2) the social needs which the school as an institution must serve; and (3) the availability and use of the material under discussion—mathematics in this case—as an instrument for such pupil development and his adjustment to such and such social needs or purposes.

The existing mathematics of the high school, and particularly that of the first two years, however taught, falls far short of satisfying the known laws of adolescent growth, and it bears little relation to any known social needs. Referring to existing mathematics, the writer of course has in mind first of all the formal algebra and geometry usually found in the first and second high-school years; and to these courses may be added

* Abstract of an article in the thirtieth year book of the National Society for the Study of Education.

the solid geometry, trigonometry, and advanced algebra commonly taught in the last two years.

Subject-Matter Must Function Throughout the Process of Learning.—You may teach the pupil much or little, but what you really teach will depend entirely upon what he can and will learn. It is simple enough to cram a youth with learning which will enable him creditably to pass off a recitation or an entrance examination. That depends upon the force and skill of the teacher. But to ground the pupil in learning which will react to the only real test, namely, "will it function?" depends as much upon the nature of the pupil's mind and the stage of his development as upon the professional tact and skill of the instructor.

There is nothing to which most processes in algebra or geometry, or indeed arithmetic, can be applied except to more algebra or arithmetic. Hence, while the pupil may for the time being attain perfect marks, his learning becomes no part of his stock of usable ideas, and he straightway forgets all about it until he is put through a naïve "review," which in its turn needs to be "reviewed" when he becomes a college Freshman or enters a shop. The first conclusion then is that we must find a kind of mathematics material not only which will function but which does function in some other field than mathematics while it is being taught, and such use must further respond to a real need felt as such at the time by the pupil. So only can mathematical concepts become realized.

Disciplinary Arguments Not a Sufficient Justification.—Unless we can find some other justification for courses in the high school, many of them will undoubtedly presently travel the road of Greek, and we shall have little that can be called education left.

It is fairly to be assumed at this day of the world that unless a course can justify itself as offering to the pupil a system of ideas which help to interpret to him his environment and enable him to react to new and strange situations in that environment, then such a course has little place in a modern educational institution.

But laying aside the purely disciplinary argument in its extreme form it may be objected, with reference to geometry

especially, that here is a method of thought in which the educated man should be trained. The contention might be granted in part if the thinking of the modern world were done in the form of the syllogism and in mathematical terms as was once the case. The fact is that the thinking of the modern world is done mainly in inductive form and in terms derived from biology. Let us consider the pupil and find out what we know about him.

In the first place, the youth when he comes to the high school is, and has been for about two years on the average, an adolescent. If a boy, he is a clumsy, awkward chap, who has lost all the nimbleness and agility which he had three or four years ago, and is now chiefly occupied physically in keeping from "falling over himself" and in keeping out of sight. Mentally his mind is dreaming and seeing things never dreamed of before. If a girl, well a mere man had perhaps best not try to do justice to her. Probably, in her own way she is at bottom in the same state as the boy, though she can laugh, or at least giggle, it off, while he cannot.

It is the worst period for anything like drill. It is a period when new ideas, especially those of a general spiritual type, are entering the opening mind in hosts of new forms; when the physical organism is undergoing a process of complete reorganization and readjustment; and when mental attitudes and powers are undergoing a similar and corresponding change.

Now, it does not at all follow that the adolescent boy or girl is incapable of mathematical concepts, or necessarily finds them distasteful. A prominent characteristic of the mental attitude of the adolescent is an openness to entirely new types of ideas as well as an entirely new set of reactions. It is very likely true that there is in the adolescent mind a capacity for apprehending new mathematical concepts of a much higher order than has generally been thought possible. In any case, it does seem to be true that the ability of the mind to apprehend new ideas is related to the ideas already actually in mind, and the ability to assimilate new notions and make them a part of the intellectual capital is largely a question of opportunities for such ideas to function in the interpretation of some feature of the environment.

The difficulty with the present high-school mathematics, especially algebra, is not in the intrinsically abstruse character of the concepts, but rather (1) in the extreme difficulty of finding an opportunity for them to function, and (2) in the fact that the algebra as taught is almost entirely an organizing and drill subject.

It isn't that the pupil is not capable of a high order of thinking; he simply hasn't had the experience with which to do his thinking. He is eager and anxious for new ideas; he never will be more so; but he cannot effectively formulate ideas which form no assimilated part of his intellectual equipment. The same boy will perform marvels of wireless telegraphy the understanding of which he has gathered from his juvenile periodical, but he will gaze stupidly at his science teacher who talks to him of the elementary units of electricity, and ultimately fail in his examination.

Fundamental Attitudes of Girls Are Even Less Favorable for Abstract Mathematics.—In our analysis of the difficulty which seems to exist in the present mathematics situation, there are important special features true in the case of girls which are not true in the case of boys. Whatever is true of the mental attitude of adolescents in general to mathematical culture, it is also true that boys are normally organized to react favorably to the functions of which mathematics must become one of the chief instruments of interpretation. The woman on the other hand is organized both bodily and mentally for dealing with an entirely different set of functions, in which mathematics plays a small part. At this particular period she must be full of new ideas and insights totally different from those which are coming to the boy of the same age.

Again the high school is conspicuously the institution in the whole course of education which is today in an unsettled state. The ancient landmarks have been torn up and the boundaries are in process of revision.

Conditions have changed. A generation ago the high school was an institution which few pupils reached. Life was relatively simple and the common school education was felt to suffice for the great majority. But since the eighties of the last century high-school enrolment has been outrunning population

in growth all over the United States. This fact points unerringly to the conclusion that the expression "common school" must be extended and applied to the secondary school. The high school is rapidly becoming on the whole not the "people's college," but a part of the educational scheme common to all.

These changes have altered greatly both the curriculum problem and the pedagogical problem in the high school.

Differentiation Now Should Fall at Beginning of Adolescenses; Not at End of Compulsory Period.—The still prevalent eighth-ninth grade division point is probably related to a process of evolution which had gradually brought about a completion of eight years of work at the average age of fourteen, when by common agreement in most states the age of compulsory education has ended. Twenty years ago the Committee of Ten foreshadowed what is rapidly coming to be seen to be a fundamental principle, namely, that the division point should come at the dawn of adolescence rather than at its most critical point. This corresponds very well to a division point between the sixth and seventh grades, which teachers have for a long time suspected to be the right one.

In Any Course, Detailed, Concrete Aims, Related to Social Needs, Must Replace Formal Aims.—The plain fact is that every school in every age has been at bottom an attempt to adjust its pupils to the requirements of the society in which they live. The state as the will of society bids us get back to the original purpose of the school, namely, getting pupils ready to live effectively in our own twentieth century United States—not in the eighteenth century, nor in Germany.

Cultural Course Related to Contemporary Needs Will Continue to be Prominent.—Your curriculum may still be strictly educative or developmental, or it may be technical with a view to immediate special training for life-work or vocation.

Mathematical Courses Should be Differentiated for Cultural and Technical Purposes.—The solution on the side of the technical high school should in principle be very simple, to wit: the thorough teaching of such processes as are needed in the industry for which training is given followed by drill to the point of efficient functioning within a narrow range.

But because, as it seems to me the facts indicate, the cultural

high school is now, and will increasingly continue to be, the type of secondary school which the American school man will have to administer and in which the majority of our secondary teachers will work, I shall deal with that type only.

Development of Adaptability in Adolescents is the Aim.—Adaptability is the standard by which all mental development above the level of tropism is to be measured.

To be more concrete, I mean that the modern American high school must produce a young man or young woman, not necessarily with specific training, but capable of intelligent adaptation in any situation in which he or she is likely to be placed.

Again, lest we forget, let it be observed that the developed capacity of the individual to react to a strange situation is a question of his possessing a working system of ideas, and not of his having exercised interminably a mythical mental faculty.

Some Elements of Knowledge Are Common to All Zones of Adaptability.—For instance, knowledge of the human body, the heritage of the race in various institutions and a racial literature, in art, in ethics, and so on. Some elements are common to two or more zones of adaptability, as, for instance, the biological sciences to the housekeeper and the agriculturist. But the specific elements which go into an understanding of the fundamental problems of the homemaker are widely different from those needed in the educational equipment of the engineer or the attorney.

One of the chief functions of the secondary school is and must necessarily be the furnishing of opportunity for a selective process to take place upon the native bent of a pupil, to discover to each so far as possible the broad zone within which his future activity will normally lie.

High-school Curricula Should be Differentiated to Parallel Broad Zones of Adult Activity.—In the larger cities this division has already been foreshadowed by the erection of distinct types of high schools, to wit: the classical high school, the high school of commerce, the mechanic arts high school, and latterly the domestic arts high school. Similarly there has recently been developed in rural communities the agricultural high school.

Then there is a picture of the well-developed program of

today in a high school, enrolling say 250 pupils, and located in the typical community of say 10,000 people with industrial interests ranging all the way from a zone of farms a few miles out to several highly developed industries in town. Such a school should offer well-differentiated curricula calculated to furnish the educational foundations for: (*a*) homemaking and housekeeping; (*b*) agriculture; (*c*) mechanical and engineering pursuits; (*d*) commerce; and (*e*) the professions through its college preparatory curriculum probably reorganized somewhat in both content and method.

There Should Be Appropriate Mathematics for Each of These Curricula.—For the girl engaged in acquiring the educational foundation for her normal life-work, but little mathematics beyond the simple arithmetical computations which she has brought with her from the elementary schools will be needed.

The mathematics of the educated farmer is, first, a good deal of practical arithmetic, but not involving any very abstruse processes; second, a good conception of the properties of plane and solid figures; third, plane trigonometry and surveying. All of these the student will use in his studies and of them he will make frequent use in his vocation.

In commerce, arithmetic and certain of the processes of algebra applied to the solution of practical commercial problems will be needed.

In the mechanic arts there is an extremely interesting field for much more mathematics than we now commonly find in the secondary school. Arithmetic enough the boy already has. He needs algebra enough to understand more useful processes, and will use constantly a considerable range of constructional geometry. But more than that, his work will give him a concrete basis for trigonometry and the elements of calculus, the latter a perfectly feasible high-school subject when taught in connection with other studies and with shop work in which it has a constant opportunity to function, as I shall attempt to show later.

The mathematics of the college-preparatory curriculum will of course relate itself to mathematics in the college, until colleges conclude to relate their mathematics to what can be done in the preparatory school.

Special Provision May be Made for Brilliant Students of Mathematics.—Probably, from 1 to 1½ per cent. of all high-school students, take them as they come, have some incipient talent of this type. All large high schools, say those enrolling 500 or more, should, I believe, provide special courses for divisions of these people permitting them in every way to fulfil their bent. In other smaller schools, it is a pretty poor teacher who will not gladly put in extra time with these brilliant minds.

There remain two other considerations related to the social purpose of the school as an institution which must be considered.

Moral Purposes.—The first of these is the moral purpose of the school. Mathematics particularly has been thought to have a special moral or quasi-moral purpose in the school on account of its excellent adaptation to disciplinary ends.

But Mathematics as Such Can Contribute Little to Moral Training.—For essential morality is I think a question of the relations of individuals in society, and all we mean by moral education is the adjustment of the pupil to the standards of life in society sanctioned by the highest social ideals of his time. And this is not a matter of book learning, but rather arises, if it arises at all, from the interaction of the various personalities composing the school, especially of course from the reactions of the pupil to the personality of his teachers.

There is one more precious feature of the mathematics of the secondary school, namely, the use of mathematics for "molding the mind of the pupil in exact methods of thinking" which is the citadel of the disciplinary position. It has been, I think, amply demonstrated that the mind which has been molded to the method of mathematics will use that method in mathematics, and in thinking allied to mathematics, alone. The mathematician himself behaves in about the same manner as other mortals in a social or a political situation, but he reacts more efficiently in a certain type of scientific situation than does he who is devoid of mathematical training. The "method of mathematics" is a highly desirable asset to certain types of education, but the method will certainly not be acquired through a period of abstract study of algebra and geometry. It can only be acquired through the constant functioning of the mathematical processes learned, in the interpretation and solution of problems presented by other subjects.

Critical Evaluation of Material Available in Mathematics.

It has already been stated that in the analysis of any program problem presented by the school, three factors must be considered. First, the nature of the pupil must be known. Second, the general purpose of the school as a social institution must be investigated. Third, the availability of the material under discussion must be criticized.

What has mathematics to offer which is essential and valuable in the process of enabling pupils to interpret new and strange situations in which they will be placed.

The Use of Mathematics as a Tool in Scientific Thinking is Most Important.—First of all, mathematics like language is in the main a "tool subject," and not one possessing inherent value of its own. There are perhaps few better criteria of the trained mind than its distinguished ability to use mathematics as an instrument for the mastery of scientific truth. The educated man endeavors to reduce all his important experiences with the material world to mathematical terms and thus to proceed confidently from step to step in his career. The uneducated man never knows exactly what his experience means, and proceeds by guess in the administration of his affairs, with great waste of energy and of substance and with a high percentage of failure.

Differentiated Courses are Needed to Provide Opportunities to Use Mathematics as a Tool.—In the first place, the traditional round of algebra, geometrical logic, advanced algebra, and trigonometry ought to be entirely abandoned and a fresh start made. Entirely different sets of mathematics material should be organized for domestic arts, for agriculture, for mechanic arts, for commerce, and for other new curricula or schools as they may be organized. In each of these several schools, mathematical processes should be taught only as fast as they are needed, but the need should be sought out and brought forward as well for the sake of the intellectual value of the subject under instruction as for the sake of the pedagogy of mathematics.

To take up each curriculum in turn.

In domestic arts, mathematics is needed to a greater or less

extent in dressmaking, in the study of house construction and of the apparatus of the household, in household accounts and other economic courses, and in the study of food values. But the mathematics needed nowhere reaches beyond the elementary principles of arithmetic and a moderate amount of mechanical drawing.

Geometry, Trigonometry, and Some Algebra in Agricultural Courses.—In the agricultural courses, for the measurement of their fields, in the laying-out of highways and drains, for analyzing the strength of the different members of buildings, for determining the profit of different fields and domestic animals, the pupils will need mathematics as a "tool" to enable them to read the situations presented to their intelligences. Then, what mathematics? Chiefly geometry and trigonometry and the art of making simple mathematical records and analyses; and out of these grows the need of some algebra.

The geometry which the educated farmer needs is the "earth measurer," not a system of logic. He needs an understanding and knowledge of the properties of plane and perhaps solid figures learned in exactly the same manner in which he learns the properties of soil in his soil physics. The geometry which is a study by constructive process with pencil and compass, with square and dividers, of the essential underlying principles of the science, is the geometry which will function and read out to the pupil the truths of which he feels the need.

The algebra needed, that is, the algebra which will function in this curriculum, centers around the equation.

Master Use of the Equation and Subordinate Processes.—Now to acquire facility in the use of the equation means a very considerable amount of practice, but such practice should of course consist in throwing into the form of equations statements which it is desirable to have in such form, and not in the solution of puzzles in the form of "problems" totally unrelated to experience. The competent use of the equation of course implies facility in the use of a limited number of other algebraic processes, to wit: the elementary concepts of algebra, the four fundamental processes with the shorter forms of multiplication and division, the simpler cases of factoring, the extraction of the square root (but not the cube), and probably an acquaintance with the essential principles of expressions in radical form.

At What Age Should Algebra and Geometry be Begun?—There is probably little or nothing in the way of introducing the type of geometrical study which I have described at any time after about the twelfth year, but the earlier the better. In the case of algebra, the unsettled state of the pupil beginning with about the age of twelve, culminating at perhaps the age of fifteen and fading into relatively settled conditions from that time on, makes attempts to develop facility in execution very unpromising before, let us say, about the age of sixteen.

Trigonometry enters at an entirely suitable period as now at about the age of seventeen or say in the eleventh or twelfth grade.

The Course Should Provide Constant Practice in the Mathematics of Records.—In the agricultural curriculum, and to a much greater extent in the commerce curriculum, is the opportunity and need of what may perhaps be called the mathematics of records. As pointed out before a characteristic difference between the educated and the uneducated man is the extent to which the former reduces his experiences to mathematical form, and reads their meaning in mathematical terms. The pupil should certainly be familiarized from the beginning of the secondary period with the practice of graphic expression and the reading of graphs.

Commercial Curriculum Presents Special Problems and Opportunities in Securing Educative Content.—A curriculum in commerce definitely and seriously organized for any purpose more worthy than as a temporary abiding-place for pupils of small ability will certainly offer broad scope for much mathematics—for more of the higher mathematics as now taught probably than any of the other curricula. This becomes at once evident when we contemplate what is involved in the rational interpretation of statistics, in the records of complicated transactions, in the understanding of banking, currency, and kindred questions which must of necessity be matters of daily experience to every really educated business man—not merely the occasional financier, but every small trader as well who would conduct his business intelligently.

The mathematics demanded by the situation and teachable in the secondary school appear to the writer to be at least the

following: (1) the science of accounts; (2) the principles of statistics; (3) the properties of number as set forth in the higher arithmetic and algebra.

Science of Accounts.—Bookkeeping, in the hands of a competent teacher, even now satisfies more of the requirements of the educative process than most high-school courses.

Principles of Statistics is Necessary.—The business man must read a very considerable part of his literature in statistical form.

To mention subjects like insurance, returns on investments, annuities, and similar considerations is to justify the need of the third kind of mathematics mentioned above, namely, the higher arithmetic and algebra or the study of the properties of number as such.

One final study is perhaps here worth while, to wit: a brief survey of the mathematics indicated for courses in the mechanic arts, for this curriculum certainly offers the broadest scope for mathematics teaching, though in the mind of the present writer commerce promises distinctly the greater intensiveness.

The mechanic arts curriculum ordinarily embraces: woodworking of a somewhat advanced type; forging; pattern-making; moulding and casting; and general machine-shop practice with the engine lathe, drill press, bed planer, and milling machine. To these must be added mechanical drawing of a more advanced type than that found elsewhere.

Now the underlying mathematics which will interpret the subject-matter of this curriculum and which will function pedagogically during the teaching process and which is probably assimilable during the secondary period is the following: (1) geometry, plane, solid, and descriptive; (2) the elements of plane trigonometry; (3) the elements of analytic geometry and calculus. Of course as mathematical tools there must also be the modicum of algebra which is really needed for reading purposes; and acquaintance with the manipulation of such devices as the slide rule and logarithmic and other tables, but not necessarily any great facility in the use of these appliances.

To a very considerable extent the educational level of the mechanic arts course will probably depend upon the degree to which the school succeeds in bringing mathematics to bear upon

the study of the various operations involved, especially in the machine shop in the later courses of the curriculum.

When the problem of the reconstruction of our mathematics is approached from the standpoint of analysis of the situation in a scientific spirit, much as has been the case with the scientific management people in the field of industry, a wonderful saving of time and a most fortunate enhancing of teaching efficiency is very likely to be the result.

SUMMARY.

The conclusions of this paper may be summarized in the following terms.

1. The traditional round of mathematics in the high school, to wit: elementary algebra, plane and solid geometry, trigonometry, and advanced algebra, must be revised both as to organization and content, and adapted to the known nature of the adolescent and to the social purpose of the high school as that purpose is increasingly revealed by modern conditions.

2. Mathematics must be treated primarily as a language, the purpose of which is the interpretation of the various sciences.

3. Courses in mathematics must be arranged at such points in the curriculum as will give immediate opportunity for functioning.

4. The several integral parts of the program such as the household arts, etc., must each have its own specially organized mathematics; and the mathematics of each curriculum should probably be in charge of the specialists of that curriculum rather than in the hands of a separate mathematics faculty.

CONCORD, N. H.

RADICALS FOR THE FRESHMAN.

By Miss Anna R. Liden.

The course in radicals, here outlined, does not pretend to offer anything new or startling. It merely suggests one way in which the freshman may be led to walk with some degree of confidence through a subject which is apt to seem to him a blind and useless maze of algebraic trickery. The work required of him is limited to what may be necessary in solving and testing equations of the quadratic form, in solving simple radical equations, and in evaluating formulæ.

Before attacking the subject of irrational quantities, the freshman reviews the definition of and conceptions concerning the root of a quantity, taking the square or cube roots of small arithmetic numbers which are perfect squares or cubes respectively, and reviewing the exponent rule for taking the root of a monomial, which was previously evolved in connection with factoring. Throughout this course, by the way, the term "root" and not "principal root" is used, for no mention is made of the possible imaginary roots.

He then passes quickly to the square root of polynomial perfect squares. This work he does primarily as a basis for taking the square root of numbers. He is led to see that an arithmetic number may be treated as a polynomial, so that the very rule used in algebra may be made to apply in arithmetic. He acquires proficiency in this work by finding the lengths of the sides of right triangles, diagonals of rectangles, sides of squares, whose areas are given, etc., the given numbers being carefully expurgated, at this stage, by means of the $3:4:5$ relation between the sides of a right triangle, so that he need take the square roots of perfect squares only.

Exponents other than positive integers come next for his consideration. Assuming that the exponent rules already learned always hold true, he develops meanings for the fractional zero, and negative exponents, agreeing with that assump-

tion. He has a little practice in the use of these exponents, principally because it simplifies the work in surds that follows.

Up to this point, he has considered rational quantities only. He now comes to irrational expressions. Would they had a more inviting name! They present difficulties enough, without being thus branded.

To begin, he considers some simple surd, $\sqrt{2}$ for instance. "There is no such thing," he says. Thereupon, he is led back to the right triangle, with which he has worked before. Using *one* for each leg, he finds the hypotenuse to be $\sqrt{2}$. Given varied values for the legs, he finds a number of surds representing lengths of lines, which soon convinces him that a surd *may be*, at any rate, a perfectly definite quantity and is therefor not nearly so absurd as it sounds. He then finds the approximate value of $\sqrt{2}$, as a decimal. "If it is a perfectly definite quantity, why can it not be expressed definitely, as a decimal?" In answer, he is told to evaluate the common fraction $\frac{1}{3}$ as a decimal. He soon decides that the difficulty is not with the surd or common fraction, but lies in the limitations of the decimal system. If, then, a surd is a definite quantity, which may appear in his formulæ and quadratic equations, he must know how to treat it.

First, he encounters the three ways in which a radical may be simplified, to facilitate handling it, and to these he gives a large part of the time and attention he puts on radicals.

From his previous study of the meaning of fractional exponents, he writes $\sqrt{ab}=(ab)^{\frac{1}{2}}=a^{\frac{1}{2}}b^{\frac{1}{2}}=\sqrt{a}\ \sqrt{b}$. Given $\sqrt{36}$, he says at once, "6." Can 36 be divided into two factors, whose roots might be taken separately, as in the preceding illustration? Yes. $\sqrt{9}\cdot\sqrt{4}=3.2=6$, the same result as before. He is then given a radicand having only one perfect square factor

$$\sqrt{50}=\sqrt{25}\cdot\sqrt{2}=5\sqrt{2}.$$

One freshman is set to work to find the approximate value of $\sqrt{50}$, while another finds the value of $5\sqrt{2}$,—results, the same. "What is the use of simplifying, when the approximate value may be obtained as easily, without simplifying?" Suppose a number of lines, represented by surds, were to be computed, as,

for instance, $\sqrt{12}$, $\sqrt{27}$, $\sqrt{48}$, $\sqrt{75}$, etc. There are four square root examples to be done. By simplifying each, $2\sqrt{3}$, $3\sqrt{3}$, $4\sqrt{3}$, $5\sqrt{3}$, one square root example results and the work is done in about one third the time. Other reasons for simplifying become obvious as he proceeds. This work is limited to the square and cube roots and simpler cases of higher degree. To increase his efficiency in handling this first method of simplifying radicals, he learns the squares of numbers up to 25 and the cubes to 10. He is also encouraged but not required to learn the decimal values for $\sqrt{2}$ and $\sqrt{3}$.

In attacking radicals with fractional radicands, he is guided somewhat after this fashion:

$$\frac{2}{5}\cdot\frac{2}{5}=\frac{4}{25}. \quad \therefore \sqrt{\frac{4}{25}}=\frac{2}{5} \text{ or } \frac{\sqrt{4}}{\sqrt{25}};$$

$$\therefore \sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{\sqrt{4}}=\frac{\sqrt{3}}{2}=\frac{1.732}{2}=.866;$$

$$\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}=\frac{1.414}{1.732}.$$

This division is performed. Now, by simplifying first thus

$$\sqrt{\frac{2}{3}\cdot\frac{3}{3}}=\sqrt{\frac{1}{9}}\cdot\sqrt{6}=\frac{\sqrt{6}}{3}=\frac{2.449}{3},$$

the last operation can be done mentally. It takes only two or three such instances to convince him of the advantage of removing the fraction from under the radical sign, before computing. After the first few examples, he does the work of simplifying under one radical sign.

He next has a very little work in lowering the order of a surd, first writing the expressions with fractional exponents and reducing the fractions to lowest terms; later, when the principle is clear, dividing the index and exponents by any common factor.

He applies these methods of simplifying radicals in evaluating formulæ such as:

$b = \sqrt{ac}$, find the value of b,
$c = \sqrt{a^2 + b^2}$, find value of c,
$A = \sqrt{s(s-a)(s-b)(s-c)}$, find value of A,
$P = 2\pi\sqrt{\frac{h}{g}}$, find value of P,

etc., writing results in simplest form first, then finding the decimal values of a few.

Then he solves incomplete and complete quadratic equations, involving radicals, such as:

$$4A^2 = 7, \text{ solve for } A,$$
$$A = \pi R^2, \text{ solve for } R,$$
$$s = \tfrac{1}{2}gt^2, \text{ solve for } t,$$
$$W = \frac{Q}{2c}, \text{ solve for } Q,$$
$$d = vt + 16t^2, \text{ solve for } t,$$
$$R = \frac{s^2 + h^2}{2h}, \text{ solve for } h,$$

etc., and problems involving formulæ and equations of the same type. The following will suffice for illustration.

1. How long will it take a body to fall 1,280 ft.? ($s = 16t^2$.)

2. The foreman of a shop reads in his book of instructions that the safe load (l), in pounds, that can be hoisted by a rope, c-inches in circumference, is found by the formula, $l = 100c^2$. How big round must his rope be to lift 500 lbs. safely?

3. A tinsmith wishes to make some cylindrical gallon cans. They are to be ten inches high. What radius must he use in drawing the base, if $V_c = \pi R^2 h$ and 1 gal. = 231 cu. in.

4. The horse power of a gasoline engine is computed by the formula

$$H = \frac{D^2 N}{2.5},$$

where H = horse power, D diam. of cylinder in inches, N = number of cylinders.

In an engine of one cylinder, find the diameter of the cylinder required to yield 40 H. P.

He does a little work in addition and subtraction, to be used later in connection with multiplication, radical equations, and checking work in quadratic equations.

He works a little with multiplication and division of radicals of the same order or of differing ones of low degrees and less still with involution and evolution. The principles are developed by means of fractional exponents. In practice, however, he does most division by rationalization, to which he gives rather more attention, limiting the work, however, to fractions having, as denominators, monomial surds of the second or third degree or binomial surds of the second. To encourage rationalization before computing the values of fractions as decimals, he is given a few fractions which are greatly simplified thereby and is required to compute them in both ways.

This work is applied in such formulæ as

$s = R\sqrt{3}$, solve for R correct to 2 places,
$h = \frac{1}{3} b\sqrt{3}$, solve for b,
$a = \frac{1}{4} s^2\sqrt{3}$, solve for s,
$s = 2a(\sqrt{2} - 1)$, solve for a,

etc.

The freshman finishes his work in surds by solving simple radical equations, testing results to be sure he has not introduced roots by squaring. He is shown the possibility of that by some such simple device as this. If $x = 2$, squaring $x^2 = 4$, writing the second equation $x^2 - 4 = 0$ or

$$(x + 2)(x - 2) = 0,\ x = 2 \text{ or } -2.$$

Throughout the course the aim is to make the freshman feel that any algebraic manipulation required of him is the shortest road to a desired goal, and not merely a devious path for the sake of the journey.

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DECOMPOSITION INTO PARTIAL FRACTIONS.

By Geo. F. Metzler.

I believe the following method of decomposition will commend itself to any who will use it enough to become familiar with its application.

A few examples will manifest its brevity.

(1) Let

$$\frac{1}{(x^2+1)(x^2+x+1)} \equiv \frac{ax+b}{y} + \frac{cx+d}{z},$$

where

$$y \equiv x^2+1, \quad \text{and} \quad z \equiv x^2+x+1.$$

Multiply both sides of (1) by y, then

$$\frac{1}{x^2+x+1} \equiv ax+b+\frac{y(cx+d)}{z},$$

which for values of x which make $y \equiv x^2+1=0$ becomes

$$ax+b \equiv \frac{1}{x} = \frac{x}{x^2} = \frac{x}{y-1} = -x;$$

therefore $a=-1$, $b=0$.

Multiply (1) by z we have

$$\frac{1}{x^2+1} \equiv \frac{z(ax+b)}{y} + cx+d;$$

then when $z=0$

$$cx+d \equiv \frac{1}{z-x} = \frac{1}{-x} = \frac{x+1}{-(x^2+x)} = -\frac{x+1}{z-1} = x+1;$$

$$\therefore\ c=1 \quad \text{and} \quad d=1.$$

(2) Again, let

$$\frac{x+1}{(x^2+1)^2(x-1)} \equiv \frac{a}{x-1} + \frac{bx+c+(dx+e)z}{(x^2+1)^2 \equiv z^2}.$$

Then

$$a = \frac{x+1}{(x^2+1)^2} = \tfrac{2}{4} = \tfrac{1}{2},$$

when $x - 1 = 0$. Therefore $a = \frac{1}{2}$.

Multiply (2) by z^2 and express the result in ascending powers of z; then

$$bx + c + (dx+e)z \equiv \frac{x+1}{x-1} - \frac{az^2}{x-1} = \frac{x^2+2x+1-a(x+1)z^2}{x^2-1}$$

$$= \frac{2x+z-a(x+1)z^2}{-2+z} = -x - \frac{x+1}{2}z.$$

$$\therefore b = -1, \quad c = 0, \quad d = -\tfrac{1}{2}, \quad e = -\tfrac{1}{2}.$$

This depends on the theorem: If an equation is an identity in x, it will become an identity in z, when it is transformed into a function of z, where z is a function of x.

Then, as we know the coefficients of z^2 and higher powers of z will equal zero, the term containing a might have been omitted in finding $bx + c + (dx+e)z$. Thus

$$bx + c + (dx+e)z \equiv \frac{x+1}{x-1} \equiv \frac{x^2+2x+1}{x^2-1} = \frac{2x+z}{-2+z} = \text{etc.}$$

It will be noticed that the numerator and denominator of $(x+1)/(x-1)$ was multiplied by $x+1$, the object being to make the denominator a function of z with coefficients independent of x.

This auxiliary factor is found thus

$$\frac{z}{x-1} = \frac{x^2+1}{x-1} = x + 1 + \frac{2}{x-1}; \ \therefore 2 + (x+1)(x-1) = z$$

or

$$(x+1)(x-1) = z - 2,$$

from which x has disappeared; then both numerator and denominator are arranged in ascending powers of z and the division performed.

(3) Let

$$\frac{x^2}{(x^2-1)(x^2+2)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+2}$$

a and b are easily found as above.

Multiply by x^2+2 and taking values that make $x^2+2=0$

$$cx+d=\frac{x^2}{x^2-1}=\frac{-2}{-3}=\frac{2}{3}\cdots.$$

(4) Let

$$\frac{x^2+1}{(x-1)^4(x^3+1)}\equiv\frac{a}{y}+\frac{bx+c}{z}+\frac{d+ew+fw^2+gw^3}{w^4},$$

where

$$y=x+1,\quad z=x^2-x+1,\quad w=x-1.$$

Multiply by z and use values that make

$$z=0 \quad\text{or}\quad x^2-x+1=0,$$

$$bx+c=\frac{x^2+1}{(x^2-2x+1)^2(x+1)}=\frac{z+x}{(z-x)^2(x+1)}$$

$$=\frac{x}{x^2(x+1)}=\frac{1}{x^2+x}=\frac{1}{z+2x-1}=\frac{1(2x-1)}{(2x-1)(2x-1)}$$

$$=\frac{2x-1}{4z-3}=\frac{2x+1}{-3}.$$

The auxiliary factor $2x-1$, which multiplies both numerator and denominator, is found by dividing $4z$ by $2x-1$.

Now multiply by w and expand in ascending powers of w we have

$$d+ew+fw^2+gw=\frac{x^2+1}{x^3+1}\cdots$$

powers of w higher than the third have coefficients equal to zero and may be omitted

$$d+ew+fw^2+gw^3=\frac{x^2+1}{x^3+1}=\frac{2+2w+w^2}{2+3w+3w^2+w^3}$$

$$=1-\frac{w}{2}-\frac{w^2}{4}+\frac{5w^3}{8}.$$

In this no auxiliary factor was needed because $x-1$ is so simple. But if the denominator were ax^2+bx+c and the

variable $w = dx^2 + rx + s$, then the auxiliary factor would be

$$d^2(ax^2 + bx + c) + (az - bd)\left(\frac{dw}{dx}\right),$$

or

$$ad^2x^2 + (2az - bd)x + cd^2 + z(az - bd),$$

and this multiplied by $ax^2 + bx + c$ gives

$$a^2w^2 + [za(cd - as) - b(bd - ar)]w + (cd - as)^2 + (cr - bs)(ar - bd)$$

which is the denominator required from which x has disappeared. This shows that the auxiliary factor may be found by using undetermined coefficients. I have never had occasion to use one so cumbersome.

Now we undertake a problem of some difficulty, in which 14 unknown quantities are to be found. We assume a knowledge of multiplication and division by detached coefficients and of how to reduce any rational function of x to a function of y where y is a rational function of x.

(5) Let

$$\frac{x^2+2}{z^2 \cdot w^4 \cdot v^2} = \frac{a+bz}{z^2} + \frac{cx+d+(ex+f)w+(gx+h)w^2+(ix+j)w^3}{w^4} + \frac{kx+l+(mx+n)v}{v^2},$$

where

$$z = x - 2, \quad w = x^2 + 1 \quad \text{and} \quad v = x^2 - x + 1.$$

Multiply by z^2 and omit z^2 and higher powers.

$$a + bz = \frac{x^2+2}{w^4 \cdot v^2} = \frac{(z+2)^2+2}{(z^2+4z+5)^4(z^2+3z+3)^2}$$

$$= \frac{6+4z+ + \cdots}{9 \cdot 5^3(5+26z+ +)} = \frac{30-136z}{9 \times 5^5}.$$

There is no need of an auxiliary factor, when z is of first degree in x.

Multiply by v^2 and omit all powers of v higher than the first.

$$kx + l + (mx + n)v = \frac{x^2+2}{z^2 \cdot w^4} = \frac{(x+1)^2(x^2+2)(x^2-2x+2)^4}{z^2(x+1)^2(w^4)(x^2-2x+2)^4}$$

$$= \frac{(x+1)^2(x^2+2)(x^2-2x+2)^4}{(v-3)^2(v^2+1)^4 = 9 - 6v + + \cdots},$$

$$\begin{array}{rl}
(x^2-2x+2)^4 = & x^8 - 8x^7 + 32x^6 - 80x^5 + 136x^4 - 160x^3 + 128x^2 - 64x + 16 \\
 & 1 \ -8 \ +32 \ -80 \ +136 \ -160 \ +128 \ -64 \ +16 \\
 & \quad 2 \ -16 \ +64 \ -160 \ +272 \ -320 \ +256 \ -128 + 32 \\
 & \qquad +1 \ -8 \ +32 \ -80 \ +136 \ -160 \ +128 - 64 + 16 \\ \hline
(x^2-2x+2)^4(x+1)^2 = & 1 \ -6 \ +17 \ -24 \ +8 \ +32 \ -56 \ +32 \ +16 - 32 + 16 \\
 & \qquad 2 \ -12 \ +34 \ -48 \ +16 \ +64 \ -112 + 64 + 32 - 64 + 32 \\ \hline
(x^2+2x+2)^4(x+1)^2(x^2+2) = & 1 \ -6 \ +19 \ -36 \ +42 \ -16 \ -40 \ +96 \ -96 + 32 + 48 - 64 + 32
\end{array}$$

Reducing this to a function of v and omitting powers higher than the first

$$\begin{array}{r|l}
-1 & 1 - 6 + 19 - 36 + 42 - 16 - 40 + 96 - 96 + 32 \ + 48 - 64 \ + 32 \\
+1 & \qquad - 1 + 5 - 13 + 18 - 11 - 13 + 38 - 45 \ + 13 + 26 \ - 35 \\
 & \quad + 1 - 5 + 13 - 18 + 11 + 13 - 38 + 45 - 13 \ - 26 + 35 \\ \hline
 & 1 - 5 + 13 - 18 + 11 + 13 - 38 + 45 - 13 - 26 \ + 35 \ \boxed{- 3x - 3} \\
 & \qquad - 1 + 4 - 8 + 6 + 3 - 16 + 19 - 10 \ - 16 \\
 & \quad + 1 - 4 + 8 - 6 - 3 + 16 - 19 + 10 + 16 \\ \hline
 & 1 - 4 + 8 - 6 - 3 + 16 - 19 + 10 + 16 \ \boxed{-20x + 19}
\end{array}$$

Divide

$$\frac{-3(x+1) + (-20x + 19)v}{9 - 6v} = -\frac{x+1}{3} + \frac{17 - 22x}{9}v$$

$$= kx + l + (mx + n)v.$$

Multiply by w^4

$$cx + d + (ex + f)w + (gx + h)w^2 + (ix + j)w^3$$

$$+ \frac{x^2+2}{z^2 \cdot v^2} = \frac{(x+2)^2(x^2+2)(x^2+x+1)^2}{(x^2-4)^2(w^2-x^2)^2},$$

$$(x^2-4)^2(w^2-x^2)^2 = (w-5)^2(w^2-w+1)^2$$

$$= 25 - 60w + 96w^2 - 82w^3,$$

$$(x^2+x+1)^2=x^4+2x^3+3x^2+2x+1,$$

$$\begin{array}{r}
1+2+3+2+1\phantom{+12+4+4+8} \\
4+8+12+8+4 \\
4+8+12+8+4 \\
\hline
1+6+15+22+21+12+4 \\
2+12+30+44+42+24+8 \\
\hline
(x^2+x+1)^2(x^2+2)(x+2)^2=1+6+17+34+51+56+46+24+8
\end{array}$$

Reducing to a function of w,

$$\begin{array}{r|l}
-1 & 1+6+17+34+51+56+46+24+8 \\
0 & -1-6-16-28-35-28-11 \\
 & +0+0+0 \\
\hline
 & 1+6+16+28+35+28+11-4x-3 \\
 & -0-1-6-15-22-20 \\
\hline
 & 1+6+15+22+20+6x+9 \\
 & -1-6-14 \\
\hline
 & 1+6+14+16+6 \\
 & -1 \\
\hline
 & 1+6+13
\end{array}$$

Then

$$\frac{-(4x+3)+(6x+9)w+(16x+6)w^2+(6x+13)w^3}{25-60w+96w^2-82w^3}$$

$$=-\frac{(4x+3)}{25}-\frac{18x+81}{125}w-\frac{568x-534}{625}w^2+\frac{7654x+1763}{3165}w^3$$

$$\equiv cx+d+(ex+f)w+(gx+h)w^2+(ix+j)w^3.$$

In like manner

$$\frac{x^2-x+1}{(x^2-x+2)^2(x^2+x+3)^3}=\frac{29+10x}{1331(x^2-x+2)^2}+\frac{22x-60}{1331(x^2-x+2)}$$

$$+\frac{242(x+1)}{1331(x^2+x+3)^3}+\frac{(4x+17)11}{1331(x^2+x+3)^2}-\frac{22x+16}{1331(x^2+x+3)},$$

(7) $$\frac{x^5-4x^4-25x^3+68x^2-13x-38}{(x-1)(x-2)^3(x^2+x+2)}=\frac{a}{x-1}$$

$$+\frac{b+cz+dz^2}{(x-2)^3=z^3}+\frac{ex+g}{y}\qquad\begin{cases}y=x^2+x+2,\\ z=x-2.\end{cases}$$

Then

$$b + cz + dz^2 = \frac{x^5 - x^4 - 25x^3 + 68x^2 - 13x - 38}{(x-1)(x^2+x+2)},$$

$$\begin{array}{r|l}
2 & 1 - 1 - 25 + 68 - 13 - 38 \\
 & \quad + 2 + \ \ 2 - 46 + 44 + 62 \\
\hline
 & 1 + 1 - 23 + 22 + 31 + 24 \\
 & \quad + 2 + \ \ 6 - 34 - 24 \\
\hline
 & 1 + 3 - 17 - 12 + \ \ 7 \\
 & \quad + 2 + 10 - 14 \\
\hline
 & 1 + 5 - \ \ 7 - 26
\end{array}$$

$$\begin{array}{r|l}
 & 1 + 1 + \ \ 2 \\
 & \quad - 1 - \ \ 1 - \ \ 2 \\
\hline
2 & 1 + 0 + \ \ 1 - \ \ 2 = (x^2 + x + 2)(x - 1) \\
 & \quad + 2 + \ \ 4 + 10 \\
\hline
 & 1 + 2 + \ \ 5 + \ \ 8 \\
 & \quad + 2 + \ \ 8 \\
\hline
 & 1 + 4 + 13 \\
 & \quad + 2 \\
\hline
 & 1 + 6
\end{array}$$

$$\frac{24 + 7z - 26z^2 + \cdots}{8 + 13z + 6z^2} = 3 - 4z + z^2 \equiv b + cz + dz^2$$

Multiply by y

$$ex + g = \frac{x^5 - x^4 - 25x^3 + 68x^2 - 13x - 38}{(x-1)(z^3)}$$

$$\begin{array}{r|l}
-2 & 1 - 1 - 25 + 68 - 13 - \ \ 38 \\
-1 & \qquad\quad - \ \ 2 + \ \ 4 + 50 - 194 \\
 & \quad - 1 + \ \ 2 + 25 - 97 \\
\hline
 & 1 - 2 - 25 + 97 - 60 - 232
\end{array}$$

$$\begin{array}{r|l}
z^3(1 - x) = & 1 - 7 + 18 - 20 + \ \ 8 \\
-2 & \qquad\quad - \ \ 2 + 16 - 48 \\
-1 & \quad - 1 + \ \ 8 - 24 \\
\hline
 & 1 - 8 + 24 - 28 - 40
\end{array}$$

$$\frac{-232 - 60x}{-40 - 28x} = \frac{(58 + 15x)(7x - 3)}{(10 + 7x)(7x - 3)}$$

$$= \frac{105x^2 + 361x - 174}{49x^2 + 49x - 30} = \frac{105y + 256x - 384}{49y - 128},$$

when powers of y are omitted this

$$= 3 - 2x \equiv g + ex, \quad 7x - 3 = \frac{49y}{7x + 10}.$$

In this case the auxiliary factor was introduced after the fraction was transformed to powers of y.

If $ax + b$ is a factor in the denominator the auxiliary factor will be $(a^2y/ax + \mathrm{b}) \ldots$, if y contains x^2.

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THE MATHEMATICIAN AND HIS LUCK.[1]

By E. A. Singer.

The mathematician of the seventeenth century is a splendid figure of pride and self-satisfaction, for he believed himself to be the sole possessor of absolute truth. The mathematician of the twentieth century is a much less splendid but much more interesting figure. Assurance has been replaced by skepticism, and satisfaction by discontent. Indeed the modern mathematician is the least happy and least contented of men,—out of sympathy with the universe, and out of humor with his luck. And that not because the gods who put the world together have treated him worse than his fellows. Just the opposite is the case. Of all possible worlds, have they not offered him just that one which is mathematically simplest, which can be described in terms of the easiest arithmetic, the easiest geometry, the easiest kinematics, and a very simple mechanics? The gods have been wonderfully good to the mathematician, and you would think that in this vale of tears he at least would be able to laugh. He is in luck, and it would seem as though he alone were in luck, for in a world as indifferent, ironical, tragic as is this one, the rest of us are not accustomed to find our ease considered, our desires satisfied. Indeed the deepest irony of all lies in the mathematical simplicity and elegance of the machine which the gods have devised for our undoing.

But is the mathematician grateful? Not he. The only lucky man in the world is unhappy just because he is in luck. He does not want his luck. He does not in his heart believe in it. He even goes so far as to suspect that there must be something wrong with himself or with his science or with his way of applying this science, that his position should be so happily anomalous. He would I think welcome any suggestion that would better his condition by making it worse and more human.

[1] Being an address read after a dinner given by the Philadelphia Section of the Association of Teachers of Mathematics of the Middle States and Maryland.

So at least I have made the mathematician out, and having made him out so, I thought there could be no more delicate and tactful way of thanking him for his hospitality on this occasion than by doing what I can to relieve him of his luck.

. . . .

The dying eighteenth century brought forth in heavy travail many new and troubling things, but none newer and more upsetting than that philosophy of Kant which Moses Mendelssohn called the all-disturbing. "A Copernican change of standpoint," was one of the expressions which Kant himself found for the revolution which had taken place within him, and how apt the phrase may be judged from a single passage of the *Kritik der reinen Vernunft* into which Kant has put all his meaning and all his daring,—"The order and uniformity of the phenomena we call nature we ourselves have put there and never should have found them had we not first put them there." The immediate followers of Kant developed this thought of the master by working out more completely the theory of *how* we put law into nature. They saw as Kant had not always seen that the imposing of such laws must be an act of the will, the expression of our choice and freedom. So that experience which had been looked upon by the English school as a mere passive reception of facts, came to be regarded as a process of construction, a work of the creative imagination. "Every fact results from an act," was Fichte's way of putting the result, and with this result whether right or wrong all later philosophy has had to reckon. Its bearing on our particular problem is obvious enough. There is no other way of freeing the mathematician from the curse of his luck than that of assuring him that he made it himself. It is this idea that I venture to put before you.

. . . .

The point of departure for the idealistic philosophy of nature was the insight that nature and science are no more capable of independent definition than are right and left, above and below, husband and wife. Nature was still the object of scientific knowledge, but only in the sense that a goal or ideal may be the object of endeavor. Solomon Maimon suggested the figure, ***limit***

of a series. As the square root of two is defined by the terms of the series that approaches it, so is nature defined by the series of ever-broadening but always finite insights which make up science in its development. Nature, we may say, is that image which science approaches as the error of observation approaches zero. So viewed, nature is no *Ding an sich* but the name of a certain ideal. Nature is completed science. Science is nature in the making.

If this analysis of meanings is accepted, one outcome is obvious,—the maker of science must be the maker of nature, and if the making of science is no mere passive receiving of hard facts, if on the contrary this making is an act, an act of the creative imagination, then surely it should not surprise us that at least some aspects of the thing made should be after our hearts' desire. This all follows *if* the scientist is permitted to use his imagination, *if* he is allowed to exercise choice. But is he? I can't help thinking that those who maintain the contrary have not lived in laboratories or observatories. They have only imagined what it would be like. They have imagined in the simplicity of their hearts that the experimenter could leave all to his experiment and do nothing himself. It would be beautiful if it were so, but no one knows better than the experimenter that his task is far from being as simple as all this comes to. He finds himself forced to choose between equally permissible images of what is going on around him. Every *laboratorius* knows this, but not every one realizes the range of choice that is open to him. He accepts many choices that have already been made as though the issue were dead. It is to the revival of these issues that a historian of the exact sciences, a Mach let us say, contributes. Moreover the statement of all the alternative images between which choice must be made, is possible only in so far as these images have actually been constructed. It is to this construction that a mathematician of the order of Poincaré contributes.

For these reasons the philosopher attaches great importance to studies which the experimental scientist is likely to view with impatience, namely the history or perhaps I should say the psychology of the sciences called exact and the activities of the mathematician constructing symbolisms that have no immediate

application. For these studies contribute more than any other to the emancipation of the scientist by actually making it possible to shatter the scheme of things and to remold it. They open the way to re-creations of nature.

. . . .

Of the choices which reflection on the history and psychology of science has brought clearly to the consciousness of the scientist, there are two which are particularly important. The first is the choice between images that are analytically connected, so that we may pass from one to the other by a recognized transformation. Such is the relation between the Ptolemaic and the Copernican pictures of the solar system. The achievement of Copernicus involved no new observation. It was truly the work of the creative imagination.

But how elementary is the transformation of coördinates within the Euclidean system when compared with the transformation of the system itself into one of the non-Euclidean orders. Yet the advantages and disadvantages of such transformation have to be exhaustively considered before we can close the issue between them. Very recently too we have come to consider transformations that involve the relative motion of various systems of coördinates. One may consider what simplicity has been introduced into our description of the phenomena of light by formulating the principle of relativity. May we not judge that we are only beginning to realize the scope and freedom of choices of this order, only beginning to understand our own omnipotence?

In the choices so far considered the alternative images are analytically connected. But there is another order of choice always forced upon the scientist that is something more than a transformation. It is the choice between the infinity of laws that fall within the probable error of his scientific data. If Copernicus illustrates the importance of transformation, Newton gives an example of a choice within limits, for surely the essential part of Newton's contribution was his ability to formulate a single law not identical with, but close enough to the separate inductions of Galileo and Kepler. The contribution of Newton as well as that of Copernicus was an act of imagina-

tion. Of course choice between limits raises an experimental issue. Between Newton on the one hand and Kepler and Galileo on the other, the choice is no longer open. But evidently as long as experimental error attaches to our data, the disappearance of an old issue is accompanied by the appearance of a new. It is for this reason that we can always treat the apparently complex as the really simple by the assumption of concealed motions. Nor can this possibility ever be eliminated. Though the range within which our choice may be exercised were reduced to the size of a pin point, this would still be a whole universe of freedom, freedom to suppose what relations we choose between the mathematical points within the pin point. To this is due the development of those images of the concealed which we so crudely and inadequately lump together under the caption of atomic theories.

. . . .

I have tried in this brief and sketchy way to suggest the sense in which the history of science is a history not merely of passive observation but also of creative imagination. We can see in what sense it is true that the heavens reflect the glory of Kepler and Newton. And can we not see as well why the mathematician is so perfectly accommodated in a world which he himself has made? I should like, if time permitted, to go further, oh, much further. I should like to raise another question, one that I dare say every one of you is asking himself at this moment,—Why, if the mathematician has made his luck, has he not made more of it? Why isn't all science as simple as geometry? Perhaps I should answer by another question, —Why since Wordsworth made such beautiful poems did he not make more of them and make them more beautiful? Perhaps because it isn't easy to create, perhaps because it is even harder to make a world than it is Columbus-like to find one already made.

But I should like to have made this vague answer more definite. I should like to have suggested that our nature-building proceeds in a certain order. It is not unlike the old fifteen puzzle, which most of you will remember. Nothing was easier than to get the first few numerals in order. Then the thing

grew harder and harder, until at the very end you generally decided that it would be better to start all over again. I mean to suggest that in the making of science all that is resistant, complex, unlovely, is left to the late-comer, and that perhaps the way of correcting this will involve beginning all over again. In which case it may well enough turn out that the geometer's ease is just the disease from which the superimposed sciences suffer, that the mathematician's luck is by way of being a human misfortune, or rather an all-too-human mistake. But to follow these hints would be to turn after-dinner into before-breakfast. Perhaps though I have said enough to suggest these two reflections,—first that in so far as the world seems made for the mathematician's ease, he has made it so himself; secondly that we are likely to need his genius to help us in making it over again. I leave him then in the hope that I have lifted a load from his shoulders. Let him not quarrel with his luck, for what is good in it isn't luck, and what is luck and accident isn't good.

University of Pennsylvania,
Philadelphia, Pa.

NEW BOOKS.

A Text-book of General Physics, Electricity, Electromagnetic Waves, and Sound. By J. A. Culler. Philadelphia: J. B. Lippincott Company. Pp. x + 311. $1.80.

This volume supplements the author's "Mechanics and Heat" already published and completes his "General Physics." The noteworthy features of the present volume are (1) the treatment of electricity and magnetism from the standpoint of the electron theory and (2) the logical development of the subject of light on the basis of the discussion of electricity. The former feature allows the author to begin his book with a very old topic, "What Electricity Is" and to follow this with some remarks on the electron theory which should be of real help to the student in forming a clear physical conception of many electrical phenomena. A similar paragraph in the second chapter, "What Magnetism Is," gives modern ideas on that subject.

While the author has shown courage in bringing back into the first pages of a modern text-book these two questions for discussion, the second feature above pointed out is a new departure that is of especial interest. Fifteen years ago the reviewer had a vision of a text-book in "General Physics" which should not be divided into mechanics, sound, heat, light, and electricity but which should be really a *general* text-book. Under force would be considered, not merely forces due to stretched strings and gravitation, but also forces due to electric fields, magnetic fields, etc.; under waves would be treated not only waves in strings but also waves in water, air, and ether. Dr. Culler has made a step in this direction. He has very well introduced the subject of light under the head of electromagnetic waves and the reviewer believes the reader of this text will have a much clearer conception of both optical and electrical phenomena as a result of this method of treatment.

Sound is disposed of in 23 pages, of which more than 4 are devoted to musical scales.

The treatment throughout is usually clear and logical, but derivations such as that of the potential due to a charged point (p. 12) are much more elegant when a little simple calculus is used. Some statements (on p. 122) with regard to the Ayrton universal shunt are misleading. The equation (94, p. 121) shows clearly that the total current, i_m, must be independent of the shunt position if the galvanometer currents are to be in the ratios 1/10, 1/100, etc. The loose statements here referred to are frequent in discussions of the Ayrton shunt.

The publishers have done a great favor to students by using a dull finished paper in the manufacture of this book. This avoids glare, which

is especially annoying in many text-books, as they are very often read by artificial light.

R. A. P.

Solid Geometry. By SOPHIA FOSTER RICHARDSON. Boston: Ginn and Company. Pp. 209. 90 cents.

Miss Richardson has written a scholarly book with some interesting features. She is evidently opposed to the tendency in some quarters to minimize the subject of solid geometry, for this book is a maximum as compared with the usual course now given in schools and colleges. It reflects a teacher who thoroughly enjoys the subject.

The incommensurable case is given unusual attention in that twenty pages of the appendix are devoted to the theory of irrational numbers and the theory of limits. This extreme treatment may well prove unfitted for the average college freshman. The book is said, however, to be just as well adapted for the entire omission of the incommensurable case. The volume of the rectangular parallelepiped is proved without using any of the theorems on the ratio between such parallelepipeds; the three dimensions being assumed commensurable with the unit in the commensurable case.

The book is explicit in stating some of the axioms of continuity and betweenness that are usually taken implicitly.

Most of the theorems are proved in full, and when proofs are omitted the reason sometimes seems to be the relative unimportance of the proposition rather than its fitness for original proof by the pupil.

On the whole the book does not seem adapted to secondary school pupils, but it will prove worth examining for those who wish a full course for college freshmen.

Vocational Arithmetic. By H. D. VINCENT. Boston: Houghton Mifflin Company. Pp. 126. 55 cents.

This book is composed of one hundred lessons on one hundred business problems, including such diverse topics as express, road building, wagon making, school financing, poultry raising, living expenses, street cars, milk industry, and cotton raising.

Each lesson contains questions of general interest regarding the industry, a list of its words to be spelled, and some requirements in writing business forms, besides the problem itself.

The problems all give practise in the use of the fundamental operations with simple numbers as they are used in the less complex operations of daily life. They might be criticised, however, in that they nearly all reduce to a credit and debit accounting of a transaction, and therefore are too much a repetition of the same methods and operations.

The book was originally written for night schools, and it seems to have a place in their work. While it is doubtful whether it would serve as an elementary school text, it has much that will repay examination by teachers in such schools.

Ann of the Blossom Shop. By Isla May Mullins. Boston: The Page Company. Pp. 308. $1.00 net.

A delightful story of the South and a sequel to "The Blossom Shop," bringing in the same character and describing the everyday life and "growing up" of some of them. It is a splendid book, especially for young people.

A Review of Algebra. By Romeyn H. Rivenburg. New York: American Book Company. Pp. 80.

This is a review book intended for a two period a week course in the senior year of the high school. It gives the various topics in very condensed form, with lists of examples of the important types. Twenty-three pages are used for college entrance examinations.

Vocational Mathematics. By William H. Dooley. Boston: D. C. Heath and Company. Pp. 341.

This book does not pretend to replace the usual work in mathematics, but is planned to supplement it by giving practice in applying its principles. It includes the commonly used parts of arithmetic, algebra, geometry and trigonometry, and seems not only to be well fitted to technical schools, but to have much of value to any teacher of mathematics.

The Teaching of Algebra (Including Trigonometry). By T. Percy Nunn. London: Longmans, Green & Company. Pp. xiv + 616. $2.00.

Exercises in Algebra (Including Trigonometry). By T. Percy Nunn. London: Longmans, Green & Company. Part I., pp. x + 356. $1.10. Part II., pp. xi + 514. $1.75.

These three books constitute one series. The author, who is professor of education in the University of London, has written a handbook of his lectures to teachers ("The Teaching of Algebra"), and to accompany this has collected exercises covering what he believes should be taught in "all stages of school instruction in the subject." The work is much more inclusive than is usual in the United States, as it contains plane and spherical trigonomery, and the elements of calculus.

The author's purpose in the teaching of mathematics is twofold: to give an understanding of the use and importance of "mathematics as an instrument of material conquests and of social organization," and to give an appreciation of the "value and significance of an ordered system of mathematics" in itself. The other aims of its teaching are all considered as comprehended in these two.

The plan of the series is unusual. In "The Teaching of Algebra" about fifty pages are given to introductory matter, and the rest of the book is divided into two parts, the first dealing with Part I. of "Exercises in Algebra," the second with Part II. In these parts, the author discusses the theory underlying the exercises under the various topics in the other books, indicates that which needs emphasis, and, in general, guides the teacher in the choice of material and the method of handling it.

There is too much of the unusual in these books for a short review to give any adequate notion of their content. It is only by careful study of them that a teacher will realize how valuable a contribution they are. Whether or not he agrees with the blending of the subjects into one whole with no distinguishable boundaries between its parts, or with the author's handling of some of the topics, any teacher will find profit in this series, and will do well to have it on his desk. It can hardly fail to add to the breadth of his viewpoint, as well as to give him some unusually good suggestions on both material and methods.

Projective Geometry. By G. B. Mathews. London: Longmans, Green and Company. Pp. xiv + 349. $1.35 net.

In order to develop the principles of projective geometry without use of the theory of distance the author follows the lead of von Staudt, Reye, and other more recent authors, leaving all reference to measurement till the latter part of the book. Without attempting a rigorous development of the elementary principles he states ten theorems and quite a number of other principles which the reader is to accept as true. Upon these he bases the thirty-two chapters, each covering briefly some phase of the subject. The principle of duality, both in the plane and in space, is introduced very early and widely used throughout the work. About the middle of the book he introduces the study of complex elements by means of elliptic involutions. After this comes quite an extended chapter on the theory of casts. Metrical and quasi-metrical properties follow. After chapters on projectivities in space, quadric surfaces, null-systems, skew involutions, line geometry, etc., he concludes with a chapter on projective problems, an extended set of exercises, and an index.

The Princess and the Clan. By Margaret R. Piper. Boston: The Page Company. Pp. 322. $1.50.

The princess is an attractive young girl from the south, who after the death of her mother and father went to live with an uncle and aunt in the north. Her governess, not pleasing the young lady, was dismissed and a young lady who was very much liked by the niece was employed. An account of her everyday life follows in which the family of the minister plays an important part.

Alma's Senior Year. By Louise M. Briefenbach. Boston: The Page Company. Pp. 318. $1.50. All those who have read the other books of the Hadley Hall Series will welcome this volume. It is a story of Alma's last year at Hadley Hall and the problems she had to meet as president of the Self-Government League. The girls had many good times and at graduation Alma and her father carried away the best part of the school.

The Proving of Virginia. By DAISY RHODES CAMPBELL. Boston: The Page Company. Pp. 340. $1.25 net.

Virginia Hammond, a young violinist with wonderful ability, meets a rich aunt at a boarding school, and although she had not been on good terms with the family since the marriage of the girl's mother to a man whom she considered beneath her, takes a great fancy to the girl and takes her home to live. Here the girl has everything that money can buy until misfortune carries away much of the aunt's property. To provide for study in Europe, Virginia in company with some other musicians tours the west and by her rare skill makes enough money to study under the great teachers of Paris.

The Spell of Tyrol. By W. D. MCCRACKEN. Boston: The Page Company. Pp. 328. $2.50 net.

It has been the aim of the author of this book to give an appreciative account of this beautiful country and its people; and this new edition of *The Fair Land Tyrol* under a new title to correspond with others of the series will undoubtedly have the good reception it deserves. The general reader of travel and description will find great interest from beginning to end, and those who have made the tour will find it refreshing and instructive. Like the other members of the series it is well illustrated.

Fundamental Sources of Efficiency. By FLETCHER DURELL. Philadelphia: J. B. Lippincott Company. Pp. 368. $2.50.

The author attempts to analyze the various sources of efficiency into a few elemental principles. Considerable has been done in studying efficiency in special fields, but such studies do not seem to have proceeded with sufficient regard to the more fundamental principles underlying all efficiency. It should prove a very useful book to all students of the problem and educators will find suggestions well worth their consideration.

The Mathematical Analysis of Electrical and Optical Wave-Motion on the Basis of Maxwell's Equations. By H. BATEMAN. Cambridge: The University Press; G. P. Putnam's Sons, New York representatives. Pp. 159. 7/6 net.

This is intended "as an introduction to some recent developments of Maxwell's eletromagnetic theory which are connected with the solution of the partial differential equation of wave-motion." The higher developments based upon dynamical equations are not considered, and, to bring the work within bounds and range of a reader with limited mathematical knowledge, some things are omitted and others stated without proof. It is a good introduction to this interesting field.

Linear Algebras. By L. E. Dickson. Cambridge: The University Press. Pp. 73. 3/– net.

This is number 16 of the *Cambridge Tracts in Mathematics and Mathematical Physics* and forms a good introduction to the general theory of linear algebras. The exposition follows that of Cartan rather than that of Molieu, Wedderburn, or Frobenius. It is an important addition to this splendid series of tracts.

A Treatise on Analytic Geometry of Three Dimensions. By George Salmon. Fifth Edition, Vol. II. Edited by Reginald A. P. Rogers. London: Longmans, Green and Company. Pp. 334. $2.25.

This volume completes the new edition of Salmon's well-known work and will be welcome to all students of advanced analytic geometry. The two volumes together form a very comprehensive algebraic and differential treatment of the subject.

The chief changes from the fourth edition consist in the addition of new articles and the rewriting of some of the old ones. In Chapter XIII. there has been added a good deal of new matter dealing with rectilinear complexes and rectilinear congruences in which Mr. Robert Russell assisted. Chapters XV. and XVI. on the cubic and quartic surfaces have been revised and enlarged, in which Mr. G. R. Webb assisted. Chapter XVII. on the general theory of surfaces has been much revised and enlarged with the help of Miss Hilda Hudson.

Complete Atlas of the World. New York: L. L. Poates and Company. Pp. 233. $1.50.

Besides splendid maps of all the countries of the world this volume contains maps of all the states and territories of the United States. There is appended a list of the counties of each state together with the principal cities and their population, and finally a list of the important cities of the world. It is a good piece of atlas making.

Tables and Formulas. Revised Edition. By W. R. Longley. Boston: Ginn and Company. Pp. 36. 50 cents.

Polyanna Grown Up. By Elenor H. Porter. Boston: The Page Company. $1.25 net.

The second *glad book* is just as charming as the first. Pollyanna growing up is just as refreshing as Pollyanna as a little girl. Several of the same characters appear again and these, with new ones, all come under the influence of Pollyanna and play the *game*.

NOTES AND NEWS.

Proposed Revision of the Constitution and By-Laws.

The Committee appointed by the Council at the meeting of November, 1914, reported to the Council and later to the Association at the meeting of April 17, 1915. The report was received and the Committee discharged. The Constitution and By-Laws by which the association is now governed were published last in the Mathematics Teacher, Vol. III, No. 3. It was voted that the new form as proposed constitute an amendment to be discussed and voted upon at the next meeting and that notice of this action be given to the members through the Mathematics Teacher.

It will be found that the new form of the Constitution and By-Laws is largely a rearrangement of the present constitution and by-laws. The few changes made are those suggested by the experience of the association. The Committee on Revision consisted of the following members: Miss Emma H. Carroll, Mr. Howard F. Hart, Miss Emma M. Requa, Mr. Ernest H. Koch, Jr., and Dr. J. T. Rorer, chairman.

Constitution of the Association of the Teachers of Mathematics in the Middle States and Maryland.

I.

Name and Object.

This Society shall be called the Association of Teachers of Mathematics in the Middle States and Maryland, and its object shall be the improvement of the teaching of mathematics.

II.

Membership.

The active membership of the Association shall consist of persons interested in the teaching of mathematics in the Middle States and Maryland, and the District of Columbia.

Honorary members, without power to vote, may be elected by the Council.

III.

Officers.

The officers of this Association shall be a President, a Vice-President, a Secretary, and a Treasurer. They shall be elected at the annual meeting, and shall hold office for one year. The President may be reëlected, but shall not hold office for more than two consecutive years.

IV.

Council.

The Council shall be composed of the officers, the chairman of the Editorial Committee, and three members at large—one to be elected at each annual meeting—and one representative from each section to be elected by the section. All members of the Council, with the exception of the officers and the chairman of the Editorial Committee, shall hold office for three years. It shall be the duty of the Council to elect new members of the Association, to arrange for all meetings of this Association, to authorize committees, and to transact all business of the Association demanding action between the annual meetings. (But it is hereby understood that this paragraph shall not be construed to legislate any one out of office until the term for which he was elected shall have expired.)

V.

Editorial Committee.

There shall be an Editorial Committee of three members whose term of office shall be three years, one to be appointed annually by the Council. The Council shall appoint one member of the Committee to serve as Chairman and as the Editor-in-Chief of the MATHEMATICS TEACHER, the official journal of the Association. The Editorial Committee shall also have charge of all the publications of this Association, with the exception of business notices sent out by the officers, but shall not have power to incur financial liability without the consent of the Council. The Editor-in-Chief shall have full power to edit and publish the

MATHEMATICS TEACHER and other publications of this Association and with the approval of the Council, shall designate such members of the Association as he shall desire to assist him in these duties.

VI.

Sections.

Whenever it shall appear to the Council that a sufficient number of members of the Association are desirous of conducting periodic meetings in any locality, the Council may authorize the formation of a section to be composed of members of the Society. The Council shall have the right to withdraw such authorization.

VII.

Meetings.

The annual meeting shall be held at a time and place to be selected by the Council.

Other meetings may be arranged at the discretion of the Council.

VIII.

Dues.

The annual dues shall be fifty cents for subscribers to the MATHEMATICS TEACHER and sixty cents to non-subscribers. All dues are payable at the time of the annual meeting.

IX.

Amendments.

Amendments to the Constitution may be proposed and discussed at any annual meeting but shall not be voted upon until the next annual meeting.

Notice of the proposed change shall be given in the call for the meeting, sent out at least two weeks in advance. Thirty members shall constitute a quorum for the consideration of a change in the Constitution, and a two-thirds vote of the members present shall be required for its adoption.

THE BY-LAWS.

I.

Payment of Dues.

1. Persons elected by the Council, as provided in Article IV of the Constitution, shall be admitted to membership in the Association upon the payment of the annual dues within sixty days of the date of their election.

2. Should the annual dues of any member remain unpaid beyond two years, after due notice his name shall be dropped from the list of members. No such person shall be eligible for re-election to membership until he shall have paid the accumulated dues up to the time his name was removed from the list of members.

II.

Program of Meeting, Order of Business, Etc.

1. The Council shall arrange the programs for the meetings of the Association.

2. The order of business at the meetings of the Association shall be as follows:

(*a*) Reading of the minutes.
(*b*) Recommendations and reports.
(*c*) Elections.
(*d*) Miscellaneous business.

3. No questions relative to administration shall be considered at any meeting except the Annual Meeting, without the recommendation of the Council.

III.

Special Meetings.

1. The President may convoke the Council whenever the affairs of the Society require it.

IV.

Nominating Committee.

1. The Council shall appoint a nominating Committee of five members of the Association to make nominations for all offices and vacancies to be filled by the Association at the next annual meeting. The Committee shall report its nominations to the

Secretary of the Association five weeks in advance of the annual meeting, at which the election is to take place, and the Secretary shall cause the nominations to be printed and sent to the members with the program of the meeting, at least one week in advance of the meeting.

V.

Appropriations to Sections.

Any section of the Association shall be entitled each year to receive from the Treasurer of the Association, for administrative and other necessary expenses, an amount not exceeding one-third of the annual dues paid by its members to the Association during the previous year, *provided*, that the Council shall have power to make additional appropriations to any section, whenever such action is deemed advisable.

VI.

Retaining Office.

All officers and chairmen of standing committees shall hold office until their successors are elected.

VII.

Amendment and Suspension of By-Laws.

No By-Law shall be enacted, amended or suspended except by a two-thirds vote of the members present at the annual meeting of the Association.

The twenty-fourth meeting of the Association of Mathematics Teachers in the Middle States and Maryland was held in Hunter College, New York City, on Saturday, April 17, 1915. The council met at 9:15 a. m. and transacted routine business. The morning session of the Meeting was called at 10:00 o'clock. Mr. Rorer reported on the revision of the constitution. Following his report the following papers were read and discussed: "The Heuristic Method; its Applications and Limitations in Secondary Mathematics," Lynn M. Saxton, College City of New York; "Corporation Schools," F. C. Henderschott, National Association of Corporation Schools; "The

Application of the Anaglyphe to the Teaching of Solid Geometry," Walter F. Shenton, The Johns Hopkins University.

The afternoon session was divided into two sections, one section on the Teaching of Algebra, in which the following were leaders of the discussion: Howard F. Hart, Montclair High School; Clarence F. Scoboria, Polytechnic Preparatory School, Brooklyn; Mabel H. Burdick, Curtis High School, New York City; the other section was on the Teaching of Calculus in which the following were leaders of the discussion: Lorraine S. Hulbert, The Johns Hopkins University; Elizabeth B. Cowley, Vassar College; William H. Metzler, Syracuse University; Herbert E. Hawkes, Columbia University; William J. Barry, Polytechnic Institute, Brooklyn.

The various papers and discussions will be published in the TEACHER.

TEACHERS' BUREAU.

C. T. 1. Man, thirty-five years old, experienced in secondary work and with nine years experience in college teaching, desires better position in college. Is teacher, not investigator.

S. T. 1. Woman, teacher of secondary mathematics, desires better position. Ph.B., 5 years experience. Excellent recommendations. Least salary $900.

S. T. 2. Woman, teacher of secondary mathematics, wishes position in a city school. A.B., 2 years, experience. Least salary $810.

G. 1. Woman, head of department in a woman's college, wishes better position. A.M., Columbia University, 8 years public school experience, 5 years private school. Can also teach Latin.

C. T. 2. Woman, substitute in college mathematics, desires a position. Ph.D., eight years experience, six in secondary schools, two in college.

S. T. 3. Woman, teacher of secondary mathematics, wishes